φ-gauge cohomology

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1 Background and motivation

Fix a prime p and a perfect field k of characteristic p. Let $W = W(k)$ and $W_n = W_n(k)$; these each come with Frobenius automorphisms, which we call σ. If X/k is a smooth proper variety, then $H^i_{\text{cris}}(X/W)$ is a finite module over W, equipped with semilinear endomorphisms F and V such that $FV = VF = p^i$. In a 2013 paper, Fontaine and Jannsen constructed a new cohomology theory that refines this structure, in a sense which we will make precise. They also indicate that their setup leads to a good setting for discussing p-torsion sheaves, which should be useful in p-adic Hodge theory.

We begin by giving the basic definitions of the categories they work in; we will come back to the motivation near the middle of the talk.

2 Gauges

2.1 Basic definitions

A *gauge* of abelian groups is a $\mathbb{Z}$-graded abelian group equipped with morphisms f of degree 1 and v of degree -1 such that $fv = vf = p$, as follows:

$$\cdots \rightleftarrows M^{-1} \rightleftarrows M^0 \rightleftarrows M^1 \cdots . \tag{1}$$

Morphisms of gauges are morphisms of degree 0 commuting with f and v. Gauges in an arbitrary abelian category, and morphisms thereof, are defined analogously. We will mostly be interested in gauges of W-module, k-vector spaces, and sheaves thereof on some site.

Remark: If R is a ring, gauges of R-modules are just graded modules over the commutative graded ring[1] $D(R) = R[f, v]/(fv - p)$, with f and v in the appropriate degrees.

*Notes for a talk in Berkeley's student arithmetic geometry seminar. Main reference: Fontaine-Jannsen, "Frobenius gauges and a new theory of p-torsion sheaves in characteristic p. I".

[1]Not graded-commutative, honestly commutative.

For $-\infty \leq a \leq b \leq \infty$, we say that a gauge M is *concentrated in the interval* $[a, b]$ if f is an isomorphism to the right of $[a, b]$ and v is an isomorphism to the left of it. In this case, M is determined up to unique isomorphism by the portion of it lying between degrees a and b inclusive.

As a simple example, a gauge concentrated in the interval $[a, a]$ has all M^i isomorphic to M^a, with $f = 1$ and $v = p$ to the right, and $f = p$ and $v = 1$ to the left.

Lemma: Let R be a noetherian ring and M a gauge of R-modules. Then the following are equivalent:

1. M is of *finite type*; i.e., is finitely generated as a $D(R)$-module.
2. Each M^i is finitely generated over R, and M is concentrated in a finite interval.

The proof is not hard, and we omit it.

2.2 Motivating example

We now pause to indicate how one can construct some interesting gauges out of something like a crystal.

Let M be a finite free W-module, and $D = M \otimes_W W[1/p]$. Suppose we are given a σ-semilinear isomorphism $\phi : D \to D$. Then we call (M, D, ϕ) a *virtual F-crystal* over k, and an F-crystal if $\phi(M) \subseteq M$. (For brevity, we will often simply say things like "M is a virtual crystal", dropping many things from the notation.)

Given a virtual crystal M, we construct a gauge as follows. For each $r \in \mathbb{Z}$, we let

$$M^r = \{m \in M : \phi(m) \in p^r M\}. \tag{2}$$

We define $v : M^r \to M^{r-1}$ to be the inclusion, and $f : M^r \to M^{r+1}$ to be multiplication by p. Clearly, $fv = vf = p$.

Since M is finitely generated, one can show that $p^b M \subseteq \phi(M) \subseteq p^a M$ for some $a \leq b$ in $\mathbb{Z}$. (Write a matrix for M, up to σ, and take the smallest and largest p-adic valuations of the entries.) It follows that $M^r = M$ for all $r \leq a$, so v is an isomorphism to the left of M^a. It's not much harder to show that f is an isomorphism to the right of M^b.[2] So in fact M is concentrated in the finite interval $[a, b]$. It is clear that each M^r is finitely generated over W, so by the lemma, M is a finite-type W-gauge.

At this point, we pause to make some more definitions. We will come back soon to add more structure to this gauge.

[2] If $r \geq b$, then $M^r = \{m \in D : \phi(m) \in p^r M\}$, since ϕ is injective. The claim follows immediately.

2.3 More definitions

Let M be a gauge in an abelian category $\mathcal{A}$. We define:

$$M^{+\infty} = \lim_{r \to \infty, f} M^r, \tag{3}$$

$$M^{-\infty} = \lim_{r \to -\infty, v} M^r. \tag{4}$$

In general, these colimits may only exist in $Ind(\mathcal{A})$, but in many cases of interest (for example, when M is concentrated in a finite interval) they are honest objects.

Fix an endofunctor $\sigma : \mathcal{A} \to \mathcal{A}$. (For us, this will always be a Frobenius twist on the category of W-modules or similar.) A *φ-module with respect to σ* is a gauge M equipped with a morphism $\varphi : \sigma(M^{+\infty}) \to M^{-\infty}$.[3] Morphisms of φ-modules are morphisms of gauges commuting with φ.

A *φ-gauge* is a φ-module in which φ is an isomorphism.

All the discussion above can be repeated with some extra structure. We will only sketch the definitions briefly, as they are technically necessary but don't add much insight. Fix a topos $\mathcal{T} = \mathrm{Sh}(\mathcal{C})$, the category of sheaves on some site. A *φ-ring* R in $\mathcal{T}$ is a commutative $\mathbb{Z}$-graded ring in $\mathcal{T}$, equipped with $f \in \Gamma(R^1)$, $v \in \Gamma(R^{-1})$, and φ satisfying the earlier relations. It is *perfect* if φ is an isomorphism.

Given a φ-ring R, there is a reasonable notion of a φ-module over it, where φ_M must be φ_R-semilinear. If R is a perfect φ-ring, a *φ-R-gauge* is a φ-module over R where φ is an isomorphism.[4]

Main example: If $1 \leq n \leq \infty$, then $W_n[f, v]/(fv - p)$ is a φ-ring (in the punctual topos $\mathcal{T} = \mathrm{Sh}(*)$), where φ is the Frobenius twist as usual. We will abbreviate this as just "W_n", as its modules are just W_n-linear gauges with semilinear φ.

Each of these categories of modules and gauges is abelian and comes with a tensor product.

2.4 Back to main example

We now return to the example of the W-gauge determined by a virtual crystal M, and endow it with the structure of a φ-gauge. For each $r \in \mathbb{Z}$, we have a map

$$\varphi_r : M^r \to M = M^{-\infty} \tag{5}$$

defined by $\varphi_r(x) = p^{-r}\phi(x)$. These are compatible along $f = p$, so they induce a map

$$\varphi : M^{+\infty} \to M^{-\infty}. \tag{6}$$

[3] If they're only Ind-objects, then such a morphism is by definition essentially a compatible family of maps from each sufficiently high-degree component to all sufficiently low-degree components.

[4] If R is not perfect, we ask instead that $\varphi'_M : R^{-\infty} \otimes_{\varphi_R, R^{+\infty}} M^{+\infty}$ be an isomorphism. For R perfect, these conditions are equivalent.

This is an isomorphism: it's injective because ϕ is; it's surjective because $\phi \otimes_W W[1/p]$ is surjective, and for any $m \in M$ we can clear the denominators of $\phi^{-1}(m)$. This discussion gives us a functor

$$\{\text{virtual crystals}/k\} \longrightarrow \{\text{finite-type } \varphi - W - \text{gauges with free components}\}, \qquad (7)$$

which one can see is fully faithful. In fact, one can do better:

Theorem: A φ-W-gauge (M, φ) is in the essential image of the functor above if and only the underlying W-gauge M is free (as a module over $D(W) = W[f, v]/(fv - p)$).

The proof takes five pages, and neither implication is trivial. We give only a very rough sketch. One shows that M is a free W-gauge if and only if M/pM is a free k-gauge; this requires a Nakayama-like lemma. The crux of the argument is to show that anything in the image of the functor is free, using another mod-p criterion which turns out to be equivalent to freeness of M/pM.

2.5 Recovering a Dieudonné module of weight i

At the beginning of this talk, I claimed that gauge cohomology would be a refinement of crystalline cohomology. I will now explain the process by which it actually recovers crystalline cohomology.

If X/k is a smooth proper variety, $H^i_{\text{cris}}(X/W)$ is a finitely generated W-module equipped with endomorphisms F[5] and V, which are respectively σ- and σ^{-1}-semilinear, such that $FV = VF = p^i$. (This follows from considering the operator $\varphi = p^i F$ on $W\Omega^i_X$ with slopes in $[i, i+1)$, chasing it through the slope spectral sequence, and using Poincaré duality.) We call such an object a *Dieudonné module of weight i*.

Given a finite-type $\varphi - W$-gauge concentrated in an interval $[a, a+i]$, say

$$M^a \rightleftarrows M^{a+1} \rightleftarrows \cdots \rightleftarrows M^{a+i-1} \rightleftarrows M^{a+i} \xrightarrow[\sim]{\varphi} M^a, \qquad (8)$$

we can recover a Dieudonné module of weight i as follows: let $M = M^a$, $F = \varphi f^i$, and $V = v^i \varphi^{-1}$.

If $i = 1$, a gauge concentrated in $[0, 1]$ contains no more information than a Dieudonné module of weight 1. But if $i > 1$ (and we ignore torsion), then a gauge concentrated in $[0, i]$ has the data of M^0, a filtration by the images of $v^r : M^r \to M^0$, and maps f and v going up and down the filtration.

Also note that nothing is stopping these objects from having p-torsion. In fact, there are examples of smooth proper X/k whose crystalline cohomology has p-torsion. Gauge cohomology will recover this.

[5] Usually called φ.

Aside: Dieudonné modules of weight 1 are usually just called Dieudonné modules. These are important in p-adic Hodge theory, as they classify certain kinds of p-group schemes over k. More specifically, there are equivalences of categories:

$$\{\text{finite-length Dieudonné modules}\}^{op} \simeq \{\text{finite commutative } p\text{-group schemes}/k\}, \tag{9}$$

$$\{\text{torsion-free Dieudonné modules}\}^{op} \simeq \{p\text{-divisible groups}/k\}, \tag{10}$$

One of Fontaine-Jannsen's motivations for writing this paper was to generalize Dieudonné theory. Roughly speaking, their goal was to define an interesting class of p-torsion sheaves on the syntomic site of a scheme X, and prove an equivalence of categories between these and some category related to gauges. The idea was that if $X = \operatorname{Spec} \mathcal{O}_K$, some p-adic integer ring, then we should have some syntomic sheaf whose generic fiber is something like μ_{p^n} (whose étale cohomology we are interested in), and whose special fiber is related to crystalline cohomology. Then one could hope that classifying such sheaves and understanding their cohomology would give rise to comparison theorems between étale and crystalline cohomology. In the case where $\mathcal{O}_K$ is unramified over $\mathbb{Z}_p$, this was essentially accomplished in the unramified case by the earlier paper of Fontaine-Messing.

3 Topologies

We will be working with sheaves on the syntomic site, which is finer than the étale or quasi-étale site but coarser than the flat (fppf) site. We now make the relevant definitions.

Definition: A morphism $\pi : X \to S$ of schemes is *syntomic*[6] if it is flat, finitely presented, and its fibers are local complete intersections; i.e. locally of the form $\operatorname{Spec} \kappa[x_1, \dots, x_n]/(f_1, \dots, f_c) \to \operatorname{Spec} \kappa$ where the source has the minimal possible dimension $n - c$.

Remark: étale $\implies$ syntomic $\implies$ flat. In characteristic p, a p-th root extraction is syntomic but not étale.

We will work on the following site. The underlying category $\mathcal{C}$ is the category of syntomic k-schemes, and the covers are surjective families of syntomic morphisms. (The paper works somewhat more generally: they consider all "absolutely syntomic" schemes, with the same topology.)

4 The gauges $\mathcal{G}_n$

The reason for using the syntomic topology comes from the Fontaine-Messing paper mentioned earlier, which exhibits some syntomic sheaves that calculate crystalline cohomology. Namely,

[6]This term was coined by Mazur; it translates approximately into Greek as "cut together".

we define:

$$\mathcal{O}_n^{\mathrm{cris}}(X) = H^0_{\mathrm{cris}}(X/W_n), \tag{11}$$

$$\mathcal{O}^{\mathrm{cris}} = \lim_{\leftarrow n} \mathcal{O}_n^{\mathrm{cris}}. \tag{12}$$

As one might expect, these are closely related to divided power structures, and indeed they are better defined in that way. We will ignore this perspective, and simply take it as a black box that these are syntomic sheaves, that their cohomology recovers the usual crystalline cohomology of X, and that each of them is endowed with a Frobenius endomorphism φ.

We now build a φ-gauge out of $\mathcal{O}^{\mathrm{cris}}$. To do this, we squint our eyes, blocking out the sheafiness of everything, and apply our main construction from earlier on the "virtual crystal" $\mathcal{O}^{\mathrm{cris}}$:

$$\mathcal{G}^r = \text{``}\{x \in \mathcal{O}^{\mathrm{cris}} : \varphi(x) \in p^r \mathcal{O}^{\mathrm{cris}}\}\text{''} \tag{13}$$

$$= \ker(\mathcal{O}^{\mathrm{cris}} \xrightarrow{\varphi} \mathcal{O}^{\mathrm{cris}} \twoheadrightarrow \mathcal{O}^{\mathrm{cris}}/p^r). \tag{14}$$

We also have a mod-p^n version: $\mathcal{G}_n^r := \mathcal{G}^r/p^n$. The construction gives rise to semilinear maps $\varphi : \mathcal{G}_n^{+\infty} \to \mathcal{G}_n^{-\infty}$. Since $\mathcal{O}^{\mathrm{cris}}$ is stable under φ (i.e. it is not just a "virtual crystal" but a "crystal"), these gauges are concentrated in $[0, \infty]$.

Theorem: For each n, $\varphi : \mathcal{G}_n^{+\infty} \to \mathcal{G}_n^{-\infty}$ is an isomorphism.

The proof takes 9 pages and involves endowing $\mathcal{O}_1^{\mathrm{cris}}$ with the structure of an F-zip; we will say nothing about it except that it is believable given the case we discussed earlier.

5 Gauge cohomology

Let X/k be a syntomic variety. We define the level-n φ-gauge cohomology of X by

$$H^i_g(X, W_n) = H^i_{\mathrm{syn}}(X, \mathcal{G}_n). \tag{15}$$

Since G_n is a φ-gauge, it is not hard to endow this with the structure of a φ-gauge as well; the degree-r part is just the cohomology of $\mathcal{G}_n^r$.

Fontaine and Jannsen prove that gauge cohomology refines crystalline cohomology in the following sense:

Theorem: For X/k proper of dimension d:

1. The gauge $H^i_g(X, W_n)$ is of finite type, and concentrated in the interval $[0, i]$. It vanishes if $i > 2d$.[7]

2. There is a canonical isomorphism $H^i_g(X, W_n)^0 = H^i_{\mathrm{cris}}(X/W_n)$.

[7] The paper says $i \geq d$, which doesn't make sense.

3. If X is smooth, proper, and irreducible, then Poincaré duality for crystalline cohomology extends to a perfect duality of φ-W_n-gauges

$$H_g^i(X, W_n) \times H_g^{2d-i}(X, W_n) \to H_g^{2d}(X, W_n) \xrightarrow{\sim} W_n(-d), \tag{16}$$

where $W_n(-d)$ is the φ-gauge concentrated in degree d with value W_n and φ acting as σ.

The comparison to crystalline cohomology follows from Fontaine-Messing's isomorphism

$$H^i_{\mathrm{syn}}(X, \mathcal{O}_n^{\mathrm{cris}}) = H^i_{\mathrm{cris}}(X/W_n), \tag{17}$$

because $\mathcal{G}_n^0 = \mathcal{O}_n^{\mathrm{cris}}$. The other statements are proved by reducing to the case $n = 1$, understanding the resulting gauge in terms of the de Rham complex of X and its cohomology, and quoting the corresponding facts about de Rham cohomology.